POSSIBLE SIMPLIFICATIONS OF THE EQUATIONS OF A TWO-TEMPERATURE PARTIALLY IONIZED PLASMA

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In investigating the behavior of an ionized gas in electromagnetic fields use is often made of the equations of conservation of mass, momentum and energy, the equation of state, Maxwell's equations and Ohm's law relating the electric field to the current flowing in the plasma. In a homogeneous isotropic medium this relation is a simple proportionality between the current density and the electric field strength [1, 2]. In the general case it is more complex in nature. Possible forms of Ohm's law for a fully ionized one-temperature plasma were investigated in [3], and for a two-temperature plasma in [4]. Moreover, it was shown in [4] that, in general, we must take into account terms proportional to the temperature gradients in Ohm's law, and that in this case it also becomes necessary to take viscous terms into account when the electron temperature exceeds the ion temperature by a significant amount. In [5] in order to facilitate the description of a three-component one-temperature plasma the equations of motion for each component, arrived at as a result of a series of simplifying assumptions, are replaced by an equation of motion for the mixture and two diffusion equations (Ohm's laws). One Ohm's law (the relation of current density to electric field) was investigated for the case of a partially ionized gas in [6, 7], where it was assumed that the medium was inviscid and had one temperature, and, moreover, that anisotropy was not allowed for in writing down the frictional forces between components.

The present paper proposes a simplification of the equations given in [8–10] for a two-temperature plasma containing electrons, singly-ionized ions, and neutral atoms. The effect of the viscosity of the components and of thermal forces is allowed for. Particle collisions are taken to be elastic, and it is assumed that $T_e \geq T$, where $T = T_i = T_a$. In the investigation we pass from the equations of motion for each component to an equation of motion for the mixture and two diffusion equations (Ohm's laws). An investigation is made of how the possible forms of diffusion equations depend on the concentration of the medium, the parameters describing the anisotropy of the transport coefficients, etc., while the necessity of allowing for viscous terms and thermal forces is also investigated. Dimensionless criteria are given for which Ohm's laws simplify considerably (viscous terms, pressure gradients, etc. may be discarded).

1. THE SYSTEM OF EQUATIONS FOR A THREE-COMPONENT TWO-TEMPERATURE PLASMA.

The transport equations for a partially ionized multi-temperature plasma have the form [8]

$$\frac{\partial n_\alpha}{\partial t} + \operatorname{div} n_\alpha \mathbf{u}_\alpha = 0, \tag{1.1}$$

$$m_\alpha n_\alpha \frac{d_\alpha \mathbf{u}_\alpha}{dt} + \nabla p_\alpha + \operatorname{div}(\pi_\alpha - m_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha) - n_\alpha e_\alpha\left(\mathbf{E} + \frac{1}{c}\mathbf{u}_\alpha \times \mathbf{B}\right) = \mathbf{R}_\alpha, \tag{1.2}$$

$$\frac{3}{2}\frac{dp_\alpha}{dt} + \frac{3}{2}p_\alpha \nabla \mathbf{u} + \nabla \mathbf{q}_\alpha + P_{\alpha ik}\frac{\partial u_i}{\partial x_k} - m_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{F}_\alpha = 3k\sum_\beta m_\alpha m_\beta n_\alpha \tau_{\alpha\beta}^{-1}(T_\beta - T_\alpha). \tag{1.3}$$

Here e_α, n_α, m_α are the charge, density and mass, respectively, of α-type particles; c is the velocity of light, k is Boltzmann's constant, $\tau_{\alpha\beta}$ the collision time for α and β type particles; and T_α, $\mathbf{q}_\alpha$, $\mathbf{u}_\alpha$ are the temperature, heat flux and mean velocity of the α-components of the plasma. Moreover, the mean relative velocity $\mathbf{w}_\alpha = \mathbf{u}_\alpha - \mathbf{u}$ of α-type particles is introduced, where u is the mean velocity of the whole mixture. In (1.2) and (1.3) the notation

$$\frac{d_\alpha}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}_\alpha \nabla), \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}\nabla),$$

$$\mathbf{F}_\alpha = \frac{e_\alpha}{m_\alpha}\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right) - \frac{d\mathbf{u}}{dt}, \quad P_{\alpha ik} = p_\alpha \delta_{ik} + \pi_{\alpha ik}$$

was employed, where E is the electric field strength, B is the magnetic induction vector, π_α is the viscous stress tensor, and p_α the partial pressure of α-components of the plasma. It is assumed that each of the components is a perfect gas, i.e., $p_\alpha = n_\alpha k T_\alpha$.

The plasma is taken to be quasi-neutral $n_e \approx n_i$. The form of the heat fluxes and viscosity terms entering into equations (1.2), (1.3) is given in [9].

We introduce the electric current density $\mathbf{j} = -n_e e(\mathbf{w}_e - \mathbf{w}_i)$ and the ion slip velocity $\mathbf{s} = \mathbf{w}_i - \mathbf{w}_\alpha$. Then for $\mathbf{R}_\alpha$ we have

$$\mathbf{R}_\alpha = \mathbf{R}_\alpha^{(1)} + \mathbf{R}_\alpha^{(2)}, \qquad \mathbf{R}_\alpha^{(1)} = \mathbf{R}_\alpha^{(j)} + \mathbf{R}_\alpha^{(s)},$$

$$\mathbf{R}_\alpha^{(2)} = \mathbf{R}_\alpha^{(t)} + \mathbf{R}_\alpha^{(T)},$$

$$\mathbf{R}_\alpha^{(j)} = \gamma_\alpha^{\parallel}\mathbf{j}_\parallel + \gamma_\alpha^{\perp}\mathbf{j}_\perp + \frac{\gamma_\alpha^{\wedge}}{B}\mathbf{j}\times\mathbf{B},$$

$$\mathbf{R}_\alpha^{(s)} = \nu_\alpha^{\parallel}\mathbf{s}_\parallel + \nu_\alpha^{\perp}\mathbf{s}_\perp + \frac{\nu_\alpha^{\wedge}}{B}\mathbf{s}\times\mathbf{B},$$

$$\mathbf{R}_\alpha^{(t)} = \delta_\alpha^{\parallel}\nabla_\parallel T_e + \delta_\alpha^{\perp}\nabla_\perp T_e + \frac{\delta_\alpha^{\wedge}}{B}\nabla T_e\times\mathbf{B},$$

$$\mathbf{R}_\alpha^{(T)} = \vartheta_\alpha^{\parallel}\nabla_\parallel T + \vartheta_\alpha^{\perp}\nabla_\perp T + \frac{\vartheta_\alpha^{\wedge}}{B}\nabla T\times\mathbf{B}.$$

The symbols ∥ and ⊥ attached to the vectors indicate that components parallel and perpendicular to the magnetic field, respectively, are taken; γ_α, ν_α, δ_α, ϑ_α are connected with the χ_α, μ_α, λ_α employed in [9]:

$$\gamma_\alpha^{\parallel(\perp,\wedge)} = \frac{a_\alpha^{\parallel(\perp,\wedge)}}{n_e e} - \chi_e^{\parallel(\perp,\wedge)}b_\alpha^{(1)} - \chi_i^{\parallel(\perp,\wedge)}b_\alpha^{(2)} - \chi_a^{\parallel(\perp,\wedge)}b_\alpha^{(3)},$$

$$\nu_\alpha^{\parallel(\perp,\wedge)} = d_\alpha^{\parallel(\perp,\wedge)} - \mu_e^{\parallel(\perp,\wedge)}b_\alpha^{(1)} - \mu_i^{\parallel(\perp,\wedge)}b_\alpha^{(2)} - \mu_a^{\parallel(\perp,\wedge)}b_\alpha^{(3)}, \tag{1.4}$$

$$\delta_\alpha^{\parallel(\perp,\wedge)} = -\lambda_e^{\parallel(\perp,\wedge)} b_\alpha^{(1)}, \qquad (1.4)\ \text{(cont'd.)}$$

$$\vartheta_\alpha^{\parallel(\perp,\wedge)} = -\lambda_i^{\parallel(\perp,\wedge)} b_\alpha^{(2)} - \lambda_a^{\parallel(\perp,\wedge)} b_\alpha^{(3)},$$

$$a_\alpha^{\parallel} = a_\alpha^{\perp} = a_\alpha, \quad d_\alpha^{\parallel} = d_\alpha^{\perp} = d_\alpha, \quad a_\alpha^{\wedge} = d_\alpha^{\wedge} = 0,$$

$$a_i = -n_i m_e \tau_{ie}^{-1}, \quad a_a = -n_e m_e \tau_{ea}^{-1}, \quad d_i = -{}^1/_2 n_i m_a \tau_{ia}^{-1},$$

$$d_e = a_a, \quad b_i^{(1)} = -\frac{a_i}{5p_e}, \quad b_e^{(2)} = -\frac{\varepsilon a_i}{5p_i}, \quad b_e^{(3)} = -\frac{\varepsilon a_a}{5p_a},$$

$$b_a^{(1)} = -\frac{d_e}{5p_e}, \quad b_a^{(2)} = -\frac{d_i}{10p_i}, \quad b_i^{(3)} = -\frac{d_i}{10p_a}, \quad \varepsilon = \frac{m_e}{m_i}.$$

The conditions

$$\sum_\alpha a_\alpha = \sum_\alpha d_\alpha = 0, \quad \sum_\alpha b_\alpha^{(l)} = 0 \quad (l = 1, 2, 3) \qquad (1.5)$$

also hold.

In order to close the system (1.1)–(1.3) we must make use of the equation of state for each component and also Maxwell's equation.

2. ANALYSIS OF THE TRANSPORT COEFFICIENTS

In order to make practical use of the coefficients we must know the dynamics of particle collisions. To be specific, we shall assume that the particles of all components are spheres and that the collision of a charged particle with a neutral, as well as the collision of two neutral particles with each other, is described by the laws of elastic collision.* In accordance with this, we have the expressions

$$Q_{aa} = Q_{ia} = 4Q_{ea}, \qquad Q_{ea} = \pi a_0^2$$

for the geometric collision cross sections $Q_{\alpha\beta}$ for α- and β-type particles, where a_0 is the radius of the first Bohr orbit. For charged particles we shall take the interaction to be a Coulomb interaction. For known $Q_{\alpha\beta}$ the collision times $\tau_{\alpha\beta}$ are also known (formulas (2.16)–(2.18) of [9]).

We introduce the parameters $\omega_e\tau_e^*$, $\omega_i\tau_i^*$ and $\omega_i\tau_{ia}$, which characterize the anisotropy. Here ω_e and ω_i are the electron and ion cyclotron frequencies, and are equal to [5]

$$\omega_e = \frac{eB}{m_e c} = 1.76\cdot 10^7 B, \qquad \omega_i = \frac{eB}{mc} = 0.96\cdot 10^4 \frac{m_p}{m} B,$$

where m_p is the proton mass, $e = |e_e|$ is the electronic charge.

For the effective times between collisions we have [9]

$$\tau_e^{*-1} = 0.4\tau_{ee}^{-1} + 1.3(\tau_{ei}^{-1} + \tau_{ea}^{-1}),$$

$$\tau_i^{*-1} = 0.4\tau_{ii}^{-1} + 0.74\tau_{ia}^{-1} + 3\varepsilon\tau_{ie}^{-1}.$$

The parameters $\omega_e\tau_e^*$ and $\omega_i\tau_{ia}$ are connected by the relation

$$\omega_i\tau_{ia} = {}^1/_{80}\,\varepsilon^{1/2}\theta^{-1/2}[13\sqrt{2} + (13\sqrt{2} + 8)\tau]\,\omega_e\tau_e^*,$$

$$\left(\theta = \frac{T}{T_e}, \quad \tau = \frac{\tau_{ea}}{\tau_{ei}} = Q\alpha, \quad \alpha = \frac{n_i}{n_a}, \quad Q = \frac{Q_{ei}}{Q_{ea}} \approx 5\cdot 10^{10}\frac{\ln\Lambda}{T_e^2}\right).$$

Here en Λ is the Coulomb logarithm; for a plasma which is not too dense en $\Lambda \approx 10$.

We note that the parameter τ bears a close relation to the degree of ionization in the plasma. Thus $\tau \to \infty$ corresponds to a fully ionized plasma (within a real range of temperatures), and $\tau \to 0$ corresponds to a weakly ionized plasma.

It is easy to see that the parameter $\omega_i\tau_{ia}$, depending on T_e and α, may be both smaller and larger than the parameter $\omega_e\tau_e^*$, though $\omega_i\tau_i^* \ll \omega_e\tau_e^*$ always.

We shall make estimates of the friction coefficients (1.4). The three last terms in the expressions for $\gamma_\alpha^{\parallel}, \gamma_\alpha^{\perp}, \nu_\alpha^{\parallel}, \nu_\alpha^{\perp}$, as well as the coefficients $\gamma_\alpha^{\wedge}$, $\nu_\alpha^{\wedge}$ are the result of allowing for terms proportional to the relative thermal fluxes of the components in the equations of motion.

The contributions due to the thermal fluxes of ions and neutrals (third and fourth terms) in the expressions (1.4) for $\gamma_e^{\parallel}$, $\gamma_e^{\perp}$, are of the order

$$\varepsilon^{1/2}\theta^{3/2}\tau(1+\tau)^{-1} \ll 1, \qquad \varepsilon^{1/2}\theta^{-3/2}(1+\tau)^{-1} \ll 1,$$

respectively, in comparison with the contribution due to the electron heat flux (second term), so that they may be neglected to obtain

$$\gamma_e^{\parallel} = \frac{a_e}{n_e e}(1 - x), \qquad \gamma_e^{\perp} = \frac{a_e}{n_e e}\left(1 - \frac{x}{1 + \omega_e^2\tau_e^{*2}}\right),$$

$$\gamma_e^{\wedge} = -\frac{a_e}{n_e e}\frac{x\omega_e\tau_e^*}{1 + \omega_e^2\tau_e^{*2}} \qquad \left(x = \frac{1 - 3\tau}{13 + (13 + 4\sqrt{2})\tau}\right). \qquad (2.1)$$

It can easily be seen that the first term plays the chief part in these expressions.

We shall compare terms in expressions (1.4) for γ_i, γ_a.

If $\varepsilon\theta^{5/2} \ll \tau \ll \varepsilon^{-1}\theta Q$, then we may also neglect the contributions due to the ion and neutral heat fluxes in comparison with the electrons in the expressions for the coefficients γ_i, γ_a, and so the coefficients acquire the same form as the expressions for γ_e (2.1), with a_e changed to a_i and a_a, respectively. In the general case only the neutral contribution may be neglected in the coefficients γ_i. Here the contribution due to the thermal flux of the ions may be less than that for the electrons, as well as being considerably in excess of it. In expressions (1.4) for γ_a we may omit to take into account the contribution due to the ion thermal flux. The contribution due to the neutrals may be smaller than, as well as considerably in excess of the electron contribution. From analysis of expressions (2.1) we conclude that in the limiting case when $\omega_e\tau_e^* \to \infty$ the frictional coefficient $\gamma_e^{\parallel}$ is 16% larger than $\gamma_e^{\perp}$, if $\tau \to \infty$, and only 7% larger if $\tau \ll 1$. In the case of finite τ the difference of the coefficients $\gamma_e^{\parallel}$ and $\gamma_e^{\perp}$ is contained within these limits.

Knowing the coefficient*

$$\mu_e^{\parallel} = -5p_e y,$$

$$y = 10^{-1}[13 + (13 + 4\sqrt{2})\tau]^{-1} + \frac{20\varepsilon Q(1 - \theta)}{13Q + (13 + 4\sqrt{2})\alpha}\,\frac{\alpha}{1 + \alpha},$$

we can make estimates in the expressions for ν_α. Here it turns out that we can neglect the last three terms in the coefficients $\nu_e^{\parallel}$ and $\nu_e^{\perp}$ in comparison with the first, and in $\nu_e^{\wedge}$ the second and third terms. We obtain as a result

$$\nu_e^{\parallel} = \nu_e^{\perp} = a_e, \qquad \nu_e^{\wedge} = a_e y(1 + \tau)\,\omega_e\tau_e^*(1 + \omega_e^2\tau_e^{*2})^{-1}.$$

We can always neglect the second and fourth terms in the expression (1.4) for $\nu_i^{\parallel}$, $\nu_i^{\perp}$ in comparison with the first. In this case if $\theta \lesssim \varepsilon$, i.e., the electron temperature is much in excess of the ion temperature, then for $\tau \to 0$ the third term in the expressions for $\nu_i^{\parallel}$ and $\nu_i^{\perp}$ (contribution due to the ion thermal flux) exceeds the first, so that these coefficients differ from each other considerably.

If $\alpha \ll \varepsilon^{-1/2}\theta^{-1/2}$, then $\nu_a^{\parallel} = \nu_a^{\perp} = d_a$, and we may neglect the first term in expression (1.4) for $\nu_a^{\wedge}$; here it turns out that

$$\nu_a^{\wedge} \sim d_a y\theta^{-1/2}\omega_e\tau_e^*(1 + \omega_e^2\tau_e^{*2})^{-1}.$$

For $\alpha \gtrsim \varepsilon^{-1/2}\theta^{-1/2}$ the term associated with the contribution due to the thermal flux of neutrals in the coefficients $\nu_a^{\parallel}$ and $\nu_a^{\perp}$ (fourth

*In all estimates carried out in future we will make considerable use of the order of the ratios $Q_{\alpha\beta}/Q_{\gamma\delta}$. To make an estimate of these ratios in the general case we can employ the theoretical and experimental data of papers [11–14].

*V. A. Polyanskii, Transport Phenomena in a Multi-Temperature Plasma, Doctoral dissertation, Moscow State University, 1965.

term) is of the same order as the first term, and there may be a considerable difference between $\nu_a\|$, and $\nu_a\perp$. In this case only the last term remains in expression (1.4) for $\nu_a\wedge$. It is easy to see that in a weakly ionized plasma the expression for $R_\alpha^{(1)}$ simplifies when $\varepsilon\theta^{1/2} \ll \tau \ll 1$:

$$R_a^{(1)} = \frac{a_a}{n_e e}\mathbf{j} + d_a\mathbf{s}.$$

Thus, for $T_e \sim 10^4$ °K, $T_i \sim 10^3$ °K (corresponding to $\theta \sim 10^{-1}$) we may neglect anisotropy for $10^{-9} \ll \alpha \ll 10^{-3}$ in the expression for the friction force. Now if

$$\tau \ll \max\{\varepsilon^{-1/2}\theta^{1/2},\ \varepsilon^{-1/2}\theta^{-1/2}Q\},$$

(this corresponds to $\alpha \ll 10^2$ for the electron and ion temperatures indicated above), we may neglect the anisotropy of the friction force between the ion and neutral components in the expression for $R_\alpha^{(1)}$

$$\mathbf{R}_a^{(1)} = \gamma_a^{\|}\,\mathbf{j}_{\|} + \gamma_a^{\perp}\mathbf{j}_{\perp} + \frac{\gamma_a^{\wedge}}{B}\mathbf{j}\times\mathbf{B} + d_a\mathbf{s}.$$

In the general case, each component is characterized by three heat conduction coefficients. If $\omega_i\tau_i^* \ll 1$, then we have only one heat conduction coefficient $\lambda_i\| = \lambda_i\perp$ for the ions and one coefficient $\lambda_a\| = \lambda_a\perp$ for the neutrals. In the isotropic case, i.e., for $\omega_e\tau_e^* \ll 1$, likewise only one coefficient remains for the electrons $\lambda_e\| = \lambda_e\perp$ and, in addition to this, simple expressions are obtained for the friction forces

$$\mathbf{R}_a^{(1)} = \gamma_a^{\|}\,\mathbf{j} + \nu_a^{\|}\,\mathbf{s}, \qquad \mathbf{R}_a^{(2)} = \delta_a^{\|}\,\nabla T_e + \vartheta_a^{\|}\,\nabla T.$$

We can easily establish that

$$\frac{\delta_a^{\|}}{\delta_e^{\|}} = \frac{\delta_a^{\perp}}{\delta_e^{\perp}} = \frac{\delta_a^{\wedge}}{\delta_e^{\wedge}} = \frac{1}{1+\tau},$$

$$\frac{\vartheta_a^{\|}}{\vartheta_e^{\|}} \sim \frac{\alpha(1-\beta)\varepsilon^{-1/2}\theta^{1/2}}{\alpha+\beta+\alpha\theta^2(1+\alpha)} \quad \left(\beta = \frac{Q_{aa}}{Q_{ii}} \sim 10^{-10}\frac{T^2}{\ln\Lambda}\right) \tag{2.2}$$

always.

3. TRANSFORMATION OF THE EQUATIONS OF MOTION

We introduce the mean velocity **u**, instead of the velocity components $\mathbf{u}_\alpha$, the current density **j** and the ion slip velocity **s** and pass to the equation of motion for the mixture as a whole and to two diffusion equations

$$\mathbf{u} = \frac{m_e n_e \mathbf{u}_e + m_i n_i \mathbf{u}_i + m_a n_a \mathbf{u}_a}{m_e n_e + m_i n_i + m_a n_a} \approx \frac{n_i \mathbf{u}_i + n_a \mathbf{u}_a}{n_i + n_a}.$$

Here and in what follows we assume that $m_i = m_a = m$, $\varepsilon = m_e/m \ll 1$.

We can easily show that

$$\mathbf{u}_e = -\frac{1}{n_e e}\mathbf{j} + \xi_a\mathbf{s} + \mathbf{u}, \qquad \mathbf{u}_i = \xi_a\mathbf{s} + \mathbf{u},$$

$$\mathbf{u}_a = -\xi_i\mathbf{s} + \mathbf{u} \tag{3.1}$$

$$\xi_a = \frac{m_a n_a}{m_e n_e + m_i n_i + m_a n_a} \approx \frac{n_a}{n_i + n_a}, \qquad \xi_i \approx \frac{n_i}{n_i + n_a},$$

where ξ_α and ξ_i are the relative concentrations of neutrals and ions.

Adding the equations of motion (1.2), we obtain

$$m(n_i + n_a)\frac{d\mathbf{u}}{dt} = -\nabla p - \operatorname{div}\pi + \frac{1}{c}\mathbf{j}\times\mathbf{B},$$

$$p = p_e + p_i + p_a, \qquad \pi = \pi_e + \pi_i + \pi_a, \tag{3.2}$$

where p, π are the pressure and viscosity tensor of the whole mixture. Following the usual pattern we introduce the diffusion relations. We use the notation

$$m_a n_a \frac{d_a \mathbf{u}_a}{dt} - \operatorname{div} m_a n_a \mathbf{w}_a \mathbf{w}_a =$$

$$= \frac{\partial}{\partial t} m_a n_a \mathbf{u}_a + \operatorname{div} m_a n_a (\mathbf{u}\mathbf{u}_a - \mathbf{u}\mathbf{u} + \mathbf{u}_a\mathbf{u}) \equiv \mathbf{I}_a$$

and introduce the characteristic parameters: dimension L, velocity U and time $T = L/U$. Neglecting the term I_e in comparison with I_i (here we take into account the fact that $\varepsilon \ll 1$), we form the equations of motion for electrons and ions. In the relationship thus obtained we replace $\mathbf{u}_i$, and in the equation of motion for neutrals $\mathbf{u}_\alpha$, with **u**. Such a change is equivalent to neglecting terms of order $\xi_a s/T$ and $\xi_i s/T$ in comparison with the terms s/τ_{ia} and s/τ_{ai}, contained in the right-hand sides of the equations. Eliminating the term du/dt from these two expressions, we arrive at an equation connecting $\mathbf{s}_\|$, $\mathbf{s}_\perp$ and $\mathbf{s}\times\mathbf{B}$. It is not hard to find one of the Ohm's laws from this equation: the expression for $\mathbf{s} = \mathbf{s}_\| + \mathbf{s}_\perp$. We set $\mathbf{s}_\|$, $\mathbf{s}_\perp$ and $\mathbf{s}\times\mathbf{B}$ in the right-hand side of the equation of motion of the electron component and employ the first relation of (3.1) in writing down the Lorentz force. Neglecting I_e in comparison with the electromagnetic term, we obtain a generalized Ohm's law (relation between **j** and **E**).

We shall confine ourselves to the treatment of a partially ionized plasma when $\alpha \lesssim 1$. Taking (1.5) into account and omitting terms of higher order of smallness, we find that the Ohm's laws have the form

$$\mathbf{s} = r^{\|}\mathbf{j}_\| + r^{\perp}\mathbf{j}_\perp - r^{\wedge}\mathbf{j}\times\mathbf{B} + \frac{1}{d_i}(\mathbf{G} + \mathbf{K} + \mathbf{R}_a^{(2)}), \tag{3.3}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B} = \frac{1}{\sigma^{\|}}\mathbf{j}_\| + \frac{1}{\sigma^{\perp}}\mathbf{j}_\perp + \frac{1}{\sigma^{\wedge}}\mathbf{j}\times\mathbf{B} +$$

$$+ \varkappa^{\|}(\mathbf{G}_\| + \mathbf{K}_\| + \mathbf{R}_{a\|}^{(2)}) + \varkappa^{\perp}(\mathbf{G}_\perp + \mathbf{K}_\perp + \mathbf{R}_{a\perp}^{(2)}) -$$

$$- \varkappa^{\wedge}(\mathbf{G}\times\mathbf{B} + \mathbf{K}\times\mathbf{B} + \mathbf{R}_a^{(2)}\times\mathbf{B}) +$$

$$+ \frac{1}{n_e e}(\mathbf{R}_e^{(2)} - \nabla p_e - \operatorname{div}\pi_e),$$

$$\mathbf{G} = \xi_a\nabla(p_e + p_i) - \xi_i\nabla p_a,$$

$$\mathbf{K} = \xi_a\operatorname{div}(\pi_e + \pi_i) - \xi_i\operatorname{div}\pi_a. \tag{3.4}$$

After a series of simplifications in which considerable use is made of the conditions $\varepsilon \ll 1$, $\varepsilon\theta \ll 1$, $\alpha \lesssim 1$, $T_e \lesssim 10^4$ °K, the coefficients appearing in equations (3.3), (3.4) may be represented in the following form:

$$r^{\|} = \frac{\gamma_a^{\|}}{d_i},\ r^{\perp} = \frac{1}{n_e e}\left(n_e e\frac{\gamma_a^{\perp}}{d_i} + \xi_a\omega_i\tau_{ia}\right),\ r^{\wedge} = \frac{\xi_a}{c d_i},$$

$$\sigma^{\|} = \frac{n_e e}{\gamma_e^{\|}},\ \sigma^{\perp} = \frac{n_e e}{\gamma_e^{\perp}(1+\Delta)},\ \sigma^{\wedge} = \frac{c n_e e}{1 + \xi_a\omega_i\tau_{ia}},$$

$$\varkappa^{\|} = \frac{1}{n_e e}\frac{d_e}{d_i},\ \varkappa^{\perp} = \frac{1}{n_e e}\left(\frac{d_e}{d_i} + \xi_a\frac{\nu_a^{\wedge}}{d_a}\,\omega_i\tau_{ia}\right),\ \varkappa^{\wedge} = r^{\wedge},$$

$$\Delta = \xi_a^2\omega_i\tau_{ia}\frac{B}{c\gamma_e^{\perp}}.$$

4. ESTIMATE OF THE TERMS IN EQUATIONS (3.2)–(3.4)

We shall assume that in equation (3.2) the momentum term, the pressure forces and viscous forces are of an order which does not exceed that of the electromagnetic forces (otherwise we could neglect the effect of the electromagnetic forces on the medium in general), whence it follows that

$$|\nabla p| \lesssim \frac{1}{c}|\mathbf{j}\times\mathbf{B}|, \qquad |\operatorname{div}\pi| \lesssim \frac{1}{c}|\mathbf{j}\times\mathbf{B}|. \tag{4.1}$$

We note that for any $\omega_e\tau_e^*$ and $\omega_i\tau_i^*$ there is, amongst the five viscosity coefficients which characterize each component in the general case, at least one ($\eta_\alpha^{(0)}$) of order larger or equal to the order of the remaining viscosity coefficients of this component

$$\eta_e^{(0)} \sim p_e\tau_e, \qquad \eta_i^{(0)} \sim p_i\tau_i, \qquad \eta_a^{(0)} \sim p_a\tau_a,$$

where τ_α are expressed by formulas (2.6) of [9].

Comparing viscosity coefficients, we obtain

$$\frac{\eta_e^{(0)}}{\eta_a^{(0)}} \sim \varepsilon^{1/2}\theta^{-1/2}\frac{1+\alpha}{1+\tau}\alpha, \qquad \frac{\eta_i^{(0)}}{\eta_a^{(0)}} \sim \alpha\frac{1+\alpha}{1+\tau\theta^{-2}}. \tag{4.2}$$

Allowing for the fact that $\alpha \lesssim 1$, $T_e \lesssim 10^4$ °K, in the case under consideration, we conclude from (4.2) that the viscous forces in equation (3.2) are entirely determined by the neutrals, i.e., $\pi = \pi_a$, so that

$$|\operatorname{div}\pi_e| \ll \frac{1}{c}|\mathbf{j}\times\mathbf{B}|, \qquad |\operatorname{div}\pi_i| \ll \frac{1}{c}|\mathbf{j}\times\mathbf{B}|,$$

$$|\operatorname{div}\pi_a| \lesssim \frac{1}{c}|\mathbf{j}\times\mathbf{B}|, \qquad \mathbf{K} = -\xi_i \operatorname{div}\pi_a. \tag{4.3}$$

Making use of the estimates, we find that the ratio of viscous to electromagnetic forces in expression (3.3) is of the order

$$\frac{1}{r^\wedge d_i}\frac{|\mathbf{K}|}{|\mathbf{j}\times\mathbf{B}|} \lesssim \alpha. \tag{4.4}$$

Keeping (4.3) in mind, we easily see that we can always neglect the term div π_e in the last bracket on the right-hand side in Ohm's law (3.4), and that the ratios of the viscous $\varkappa^\parallel \mathbf{K}_\parallel$ and $\varkappa^\wedge\mathbf{K}\times\mathbf{B}$ to the electromagnetic terms $(1/\sigma^\wedge)\,\mathbf{j}\times\mathbf{B}$ are of the order

$$\varkappa^\parallel\sigma^\wedge\frac{|\mathbf{K}_\parallel|}{|\mathbf{j}\times\mathbf{B}|} \lesssim \varepsilon^{1/2}\theta\,[1 - y(1+\tau)],$$

$$\varkappa^\wedge\sigma^\wedge\frac{|\mathbf{K}\times\mathbf{B}|}{|\mathbf{j}\times\mathbf{B}|} \lesssim \alpha\frac{\omega_i\tau_{ia}}{1+\omega_i\tau_{ia}}. \tag{4.5}$$

Comparing transport coefficients, we obtain

$$\frac{r^\perp}{r^\parallel} \sim 1 + 10^{-1}(1+\tau)\,\omega_e\tau_e^*, \qquad \frac{r^\wedge B}{r^\perp} \sim \frac{(1+\tau)\,\omega_e\tau_e^*}{1+(1+\tau)\,\omega_e\tau_e^*},$$

$$\frac{\sigma^\perp}{\sigma^\parallel} \sim \frac{1}{1+\omega_e\tau_e^*\omega_i\tau_{ia}}, \qquad \frac{\sigma^\perp B}{\sigma^\wedge} \sim \frac{(1+\omega_i\tau_{ia})\,\omega_e\tau_e^*}{1+\omega_e\tau_e^*\omega_i\tau_{ia}},$$

$$\frac{\varkappa^\perp}{\varkappa^\parallel} \sim 1 - 10^{-1}y\frac{(1+\tau)\,\omega_e^2\tau_e^{*2}}{1+\omega_e^2\tau_e^{*2}},$$

$$\frac{\varkappa^\wedge B}{\varkappa^\parallel} \sim \frac{(1+\tau)(1+\omega_e\tau_e^*)\,\omega_e\tau_e^*}{1+\omega_e^2\tau_e^{*2} + y(1+\tau)(\omega_e\tau_e^* - 1)\,\omega_e\tau_e^*}. \tag{4.6}$$

For any relations between the partial pressure gradients the following estimates hold

$$|\mathbf{G}| \lesssim |\nabla p|, \qquad |\nabla p_e| \lesssim |\nabla p|. \tag{4.7}$$

We shall assume that

$$|\mathbf{E}| \sim \frac{1}{c}|\mathbf{u}\times\mathbf{B}|. \tag{4.8}$$

However, if $|\mathbf{E}| \gg c^{-1}|\mathbf{u}\times\mathbf{B}|$, we may omit the termm $c^{-1}\mathbf{u}\times\mathbf{B}$ in (3.4).

5. POSSIBLE FORMS OF OHM'S LAW

The estimates given in (4.4), (4.5) show that generally speaking, viscous terms must be taken into account in the Ohm's laws for a three-component plasma. We shall write out the possible simplified forms of these relations. In so doing we shall make considerable use of estimates (2.2), (4.1)–(4.8), from which we conclude that the relative magnitude of the terms in the Ohm's laws is determined by the magnitude of the following dimensionless parameters:

$$\alpha,\ \tau,\ \theta,\ \omega_e\tau_e^*.$$

We can easily show that for $\theta \gtrsim \varepsilon^{1/4}$ in the expression $\varkappa^\parallel\mathbf{R}_{a\parallel}^{(2)}$, entering into the right-hand side of (3.4), we can neglect terms proportional to the electron temperature gradient in comparison with the analogous term entering into the expression $(1/n_ee)\,\mathbf{R}_e^{(2)}$. If $\omega_e\tau_e^* \ll \varepsilon^{-1/4}\theta^{1/2}$, then the corresponding terms may also be neglected in the expressions $\varkappa^\perp\mathbf{R}_{a\perp}^{(2)}$, $\varkappa^\wedge\mathbf{R}_a^{(2)}\times\mathbf{B}$, entering into the same expression.

In further estimates we shall compare the viscous and pressure forces with the electromagnetic terms. Upon so doing, it turns out that in both Ohm's laws and in those cases when it is not necessary to take into account terms proportional to $\mathbf{j}\times\mathbf{B}$, viscous and pressure forces may also be neglected. In addition, we need not take into account viscosity and pressure gradients in expression (3.3) in a weakly ionized plasma, when $\alpha \ll 1$, and viscosity terms drop out of (3.4) for the less stringent requirement $\alpha\omega_i\tau_{ia} \ll 1$.

1°. Let $\omega_e\tau_e^* \sim 1$, $\varepsilon^{1/2}\beta \lesssim \alpha \ll 1$, $\theta \gtrsim \varepsilon^{1/4}$. If $\tau \lesssim \varepsilon^{-1/4}$, then we may neglect terms in the expression $(1/n_ee)\cdot\mathrm{Re}^{(2)}$ proportional to $\nabla_\parallel T$, $\nabla_\perp T$ $\nabla T\times\mathbf{B}$, in comparison with terms $\varkappa^\parallel\mathbf{R}_{a\parallel}^{(T)}$, $\varkappa^\perp\mathbf{R}_{a\perp}^{(T)}$, $\varkappa^\wedge\mathbf{R}_a^{(T)}\times\mathbf{B}$. Moreover, the pressure gradients of all components drop out of equation (3.3), and only the electron pressure gradient remains in (3.4).

1°. 1. If $\tau \sim \varepsilon^{-1/4}$ (for example, let $T_e \sim 10^4$ °K, $\alpha \sim 10^{-2}$), then we may assume that $\varkappa^\parallel = \varkappa^\perp$, but $\sigma^\parallel$ and $\sigma^\perp$ differ considerably from each other; the difference between $r^\parallel$ and $r^\perp$ is less significant. In both Ohm's laws anisotropy appears in the currents and thermal forces. The Ohm's laws assume the form

$$\mathbf{s} = r^\parallel\mathbf{j}_\parallel + r^\perp\mathbf{j}_\perp - r^\wedge\mathbf{j}\times\mathbf{B} + \frac{1}{d_i}\mathbf{R}_a^{(2)}, \tag{5.1}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B} = \frac{1}{\sigma^\parallel}\mathbf{j}_\parallel + \frac{1}{\sigma^\perp}\mathbf{j}_\perp + \frac{1}{\sigma^\wedge}\mathbf{j}\times\mathbf{B} + \varkappa^\parallel\mathbf{R}_a^{(T)} - \varkappa^\wedge\mathbf{R}_a^{(T)}\times\mathbf{B} + \frac{1}{n_ee}(\mathbf{R}_e^{(t)} - \nabla p_e). \tag{5.2}$$

1°. 2. If $\tau \sim 1$ (for $T_e \sim 10^4$ °K this corresponds to $\alpha \sim 10^{-3}$), then $r^\perp$ is greater than $r^\parallel$ by less than 10%, and the difference between $\sigma^\parallel$ and $\sigma^\perp$ may be several times larger than between $r^\perp$ and $r^\parallel$. In the diffusion equation (3.3) we may disregard anisotropy in the currents

$$\mathbf{s} = r^{\parallel}\mathbf{j} - r^{\wedge}\mathbf{j}\times\mathbf{B} + \frac{1}{d_i}\mathbf{R}_a^{(2)}. \tag{5.3}$$

The second relation has its previous form (5.2).

1°. 3. For $\tau \ll 1$ the anisotropy in the currents disappears in both equations. The Ohm's laws assume the form

$$\mathbf{s} = r^{\parallel}\mathbf{j} - r^{\wedge}\mathbf{j}\times\mathbf{B} + \frac{1}{d_i}\mathbf{R}_a^{(2)},$$

$$\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B} = \frac{1}{\sigma^{\parallel}}\mathbf{j} + \frac{1}{\sigma^{\wedge}}\mathbf{j}\times\mathbf{B} + \varkappa^{\parallel}\mathbf{R}_a^{(T)} - $$

$$- \varkappa^{\wedge}\mathbf{R}_a^{(T)}\times\mathbf{B} + \frac{1}{n_e e}(\mathbf{R}_e^{(t)} - \nabla p_e). \tag{5.4}$$

In cases (1°. 2), (1°. 3) we may assume that

$$r^{\parallel} = r_0, \quad \sigma^{\parallel} = \sigma_0, \quad \sigma^{\perp} = \frac{\sigma_0}{1 + \xi_a^2\omega_e\tau_0\omega_i\tau_{ia}}, \quad \sigma^{\wedge} = cn_e e,$$

$$r_0 = \frac{1}{n_e e}\frac{a_a}{d_i}, \quad \sigma_0 = \frac{n_e e^2\tau_0}{m_e}, \quad \tau_0^{-1} = \tau_{ei}^{-1} + \tau_{ea}^{-1}. \tag{5.5}$$

and the equations simplify considerably.

2°. Let $\varepsilon \ll \omega_e\tau_e^* \ll 1$, $\theta \gtrsim \varepsilon^{1/4}$. The following cases are possible.

2°. 1. If $\alpha \sim 1$, $\tau \sim \varepsilon^{-3/4}$ (this corresponds to $T_e \sim 10^{4°}$ K), then the viscosity of the neutrals and the pressure forces must be taken into account in the first relation; here anisotropy of the transport coefficients is absent

$$\mathbf{s} = r^{\parallel}\mathbf{j} - r^{\wedge}\mathbf{j}\times\mathbf{B} + \frac{1}{d_i}(\mathbf{G} - \xi_i \operatorname{div}\pi_a + \mathbf{R}_a^{(2)}). \tag{5.6}$$

The electron pressure gradient must also be omitted in the second expression, and anisotropy is also absent

$$\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B} = \frac{1}{\sigma_0}\mathbf{j} + \varkappa^{\parallel}\mathbf{R}_a^{(T)} - $$

$$- \varkappa^{\wedge}\mathbf{R}_a^{(T)}\times\mathbf{B} + \frac{1}{n_e e}\mathbf{R}_e^{(t)}. \tag{5.7}$$

Generally speaking, we may not neglect the second term on the right-hand side.

2°. 2. For $\varepsilon^{1/2}\beta \lesssim \alpha \ll 1$ and $\tau \lesssim 1$ the Ohm's laws assume the form

$$\mathbf{s} = r_0\mathbf{j} + \frac{1}{d_i}\mathbf{R}_a^{(2)}, \tag{5.8}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B} = \frac{1}{\sigma_0}\mathbf{j} + \varkappa^{\parallel}\mathbf{R}_a^{(T)} + \frac{1}{n_e e}\mathbf{R}_e^{(t)} \tag{5.9}$$

3°. If $\varepsilon^{1/2}\beta \lesssim \alpha \lesssim 1$, $\tau \lesssim \varepsilon^{-3/4}$, $\theta \gtrsim \varepsilon^{1/4}$, then the Ohm's laws reduce to the form 2°. 2, for $\omega_e\tau_e^* \lesssim \varepsilon$.

For $|\nabla_{\parallel} T| / |\nabla_{\parallel} T_e| \sim |\nabla_{\perp} T| / |\nabla_{\perp} T_e| \lesssim 1$ we may omit terms proportional to the ion temperature gradient in all the expressions which have been given.

REFERENCES

1. L. D. Landau and E. M. Lifshitz, Electrodynamics of Continua [in Russian], Gostekhizdat, 1957.
2. T. Cowling, Magnetohydrodynamics [Russian translation], Izd. inostr. lit., 1959.
3. V. B. Baranov and G. A. Lyubimov, "The form of Ohm's law in a fully ionized gas," PMM, vol. 25, no. 3, 1961.
4. V. V. Gogsosov, "Possible simplified equations of a fully ionized two-temperature plasma," PMM, vol. 28, no. 5, 1964.
5. S. I. Braginskii, "Transport in a plasma," collection Problems of Plasma Theory [in Russian], Gosatomizdat, 1963.
6. A. G. Kulikovskii and G. A. Lyubimov, Magnetohydrodynamics [in Russian], Fizmatgiz, 1962.
7. V. M. Zhdanov, "Transport phenomena in a partially ionized gas," PMM, vol. 26, no. 2, 1962.
8. M. Ya. Alievskii and V. M. Zhdanov, "Transport equations for a noniosthermal polytropic plasma," PMTF, no. 5, 1963.
9. M. Ya. Alievskii, V. M. Zhdanov, and V. A. Polyanskii, "The viscous stress tensor and thermal flux in a two-temperature partially ionized gas," PMTF, no. 3, 1964.
10. V. A. Polyanskii, "Diffusion and conductivity in a partially ionized multi-temperature gas mixture," PMTF, no. 5, 1964.
11. S. Brown, Basic Data of Plasma Physics [Russian translation], Gosatomizdat, 1961.
12. H. Massey and E. Burhop, Electronic and Ionic Impact Phenomena [Russian translation], Izd. inostr. lit., 1958.
13. J. O. Hirschfelder, C. Curtiss, and R. Bird, Molecular Theory of Gases and Liquids [Russian translation], Izd. inostr. lit., 1961.
14. A. Dalgagno, "Charged particles in the upper atmosphere," Ann. Geophys, vol. 17, p. 16, 1961.

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